

# Andoyer construction for Hill and Delaunay variables

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**Abstract** Andoyer variables are well known for the study of the rigid body dynamics. But these variables were derived by Andoyer through a procedure that can be also used to obtain the Delaunay variables of the Kepler problem without the use of Hamilton-Jacobi theory or non intuitive generating functions.

**Keywords** Celestial mechanics · Delaunay variables · Hill variables · Andoyer variables

## 1 Introduction

Since Binet's work (Binet, 1841) and Tisserand thesis (Tisserand, 1868) the classical derivation of the Delaunay variables of the elliptical Keplerian motion (Delaunay, 1860) is made through Hamilton-Jacobi theory. Although ubiquitous in celestial mechanics textbooks<sup>1</sup>, the derivation is not very natural or easy to understand for students. Here we present a derivation of the Delaunay variables that does not require Hamilton-Jacobi theory. We use the intermediate derivation of the Hill variables (Hill, 1913) following Andoyer (Andoyer, 1915, 1923). Transformations from Hill variables to Delaunay variables exist in the literature (Andoyer, 1913; Deprit, 1981; Floria, 1995) but they rely on a generating function that is not very natural to the author and which we will avoid here by using a more direct computation based on the invariance of the canonical differential 2-form.

Henri Andoyer (1862-1929) is well-known for the derivation of the Andoyer variables that are very well adapted to the rigid body dynamics (Andoyer, 1923). In fact, the derivation of these variables is obtained through a very general procedure that can be also applied to the Keplerian two-body problem, and which then leads to the Hill variables (Andoyer, 1915).

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<sup>1</sup> One can find in (Fejoz, 2013) an alternate presentation avoiding Hamilton-Jacobi theory.

## 2 Andoyer canonical criterion

We recall here the Andoyer derivation (Andoyer, 1923). Let us consider a  $n$  degree of freedom Hamiltonian system with canonical variables  $(p_j, q_j)$ , where  $q_j$  are the positions and  $p_j$  the momenta. We then make the change of variables  $(p_j, q_j) \rightarrow (y_k, z_k)$  which we assume to be a good differentiable change of variable on the domain of interest. We have

$$\sum_j p_j dq_j = \sum_{i,k} p_j \frac{\partial q_j}{\partial y_k} dy_k + \sum_{i,l} p_j \frac{\partial q_j}{\partial z_l} dz_l \quad (1)$$

For any variable  $\alpha$ , let us denote

$$J_\alpha = \sum_i p_j \frac{\partial q_j}{\partial \alpha} . \quad (2)$$

Andoyer assumes that

$$\forall k = 1, \dots, n \quad J_{y_k} = 0 \quad (3)$$

and that

$$\forall k = 1, \dots, n \quad J_{z_k} = u_k(y_l) \quad (4)$$

where  $(y_k)_{k=1, \dots, n} \rightarrow (u_k)_{k=1, \dots, n}$  is a diffeomorphism. We have then

$$\sum_j p_j dq_j = \sum_k u_k dz_k \quad (5)$$

The change of variable  $(p_j, q_j) \rightarrow (u_k, z_k)$  conserves the 1-form  $\sum_j p_j dq_j$  and thus the canonical 2-form  $\sum_j dp_j \wedge dq_j$ . It is thus canonical<sup>2</sup>. To search for such a canonical change of variable, one thus needs to compute  $J_\alpha$  for all of the new variables  $\alpha$ . Andoyer remarks then that  $J_\alpha$  is the scalar product

$$J_\alpha = \mathbf{p} \cdot \mathbf{V}_\alpha \quad (6)$$

where  $\mathbf{p} = (p_j)$  is the momentum vector, and  $\mathbf{V}_\alpha = (\frac{\partial q_j}{\partial \alpha})$  is the virtual velocity obtained when varying the only variable  $\alpha$ . Everything can then be obtained without practically any computation.

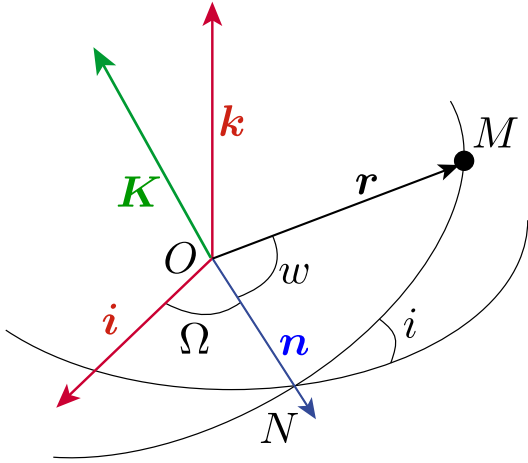
## 3 Hill variables

We consider the Kepler problem in a fixed reference frame  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ , with radius vector  $\mathbf{r} = r\mathbf{u}$ , gravitational constant  $\mu$ , and Hamiltonian

$$\mathcal{H} = \frac{1}{2} \dot{\mathbf{r}}^2 - \frac{\mu}{r} \quad (7)$$

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<sup>2</sup> The transformations conserving the 1-form  $\sum_j p_j dq_j$  form a subgroup of the canonical transformations and are called Mathieu canonical transformations (Mathieu, 1874; Whittaker, 1904)



**Fig. 1** Reference frame and notations. The vectors  $\mathbf{i}, \mathbf{k}, \mathbf{K}, \mathbf{n}$  are unitary vectors.

The orbital plane  $(\mathbf{r}, \dot{\mathbf{r}})$  is orthogonal to the angular momentum  $\mathbf{G} = G\mathbf{K}$  ( $G = \|\mathbf{G}\|$ ), and defined by the longitude of the node  $\Omega$  and inclination  $i$  (Fig.1). The position of the celestial body  $M$  is defined when  $r$  and the argument of latitude  $w = \omega + v$  ( $\omega$  is the argument of perihelion, and  $v$  the true anomaly) are given. We have thus characterized the position of  $M$  with the four variables  $(r, \Omega, w, i)$ . With Hamiltonian (7), the momentum vector is simply the velocity  $\dot{\mathbf{r}}$ . Following Andoyer, we will extend the transformation  $(\dot{\mathbf{r}}, \mathbf{r}) \rightarrow (r, \Omega, w, i)$  into a true canonical change of variables. For this, we need to evaluate all virtual velocities  $\mathbf{V}_\alpha$  and  $J_\alpha = \dot{\mathbf{r}} \cdot \mathbf{V}_\alpha$  quantities (6). We remind that when rotating a vector  $\mathbf{u}$  around a fixed unitary vector  $\mathbf{k}$  with an angle  $\theta$ , we have

$$\frac{d\mathbf{u}}{d\theta} = \mathbf{k} \wedge \mathbf{u} . \quad (8)$$

The virtual velocity  $\mathbf{V}_i$  is obtained from a rotation of angle  $i$  around  $\mathbf{n}$ , the unit vector in the direction of  $\mathbf{ON}$ . We have thus

$$\mathbf{V}_i = \mathbf{n} \wedge \mathbf{r} ; \quad J_i = \dot{\mathbf{r}} \cdot \mathbf{V}_i = 0 . \quad (9)$$

In a similar way, we have

$$\begin{aligned} \mathbf{V}_w &= \mathbf{K} \wedge \mathbf{r} ; & J_w &= \dot{\mathbf{r}} \cdot (\mathbf{K} \wedge \mathbf{r}) = \mathbf{K} \cdot (\mathbf{r} \wedge \dot{\mathbf{r}}) = G ; \\ \mathbf{V}_\Omega &= \mathbf{k} \wedge \mathbf{r} ; & J_\Omega &= \dot{\mathbf{r}} \cdot (\mathbf{k} \wedge \mathbf{r}) = \mathbf{k} \cdot (\mathbf{r} \wedge \dot{\mathbf{r}}) = G \cos i . \end{aligned} \quad (10)$$

and as  $\mathbf{r} = r\mathbf{u}$

$$\mathbf{V}_r = \mathbf{u} ; \quad J_r = \dot{\mathbf{r}} \cdot \mathbf{u} = (\dot{r}\mathbf{u} + r\dot{\mathbf{u}}) \cdot \mathbf{u} = \dot{r} . \quad (11)$$

Moreover, as  $\mathbf{r}$  depends only on  $r, w, \Omega, i$  and not on  $\dot{r}$  or  $G$ , we have  $\mathbf{V}_{\dot{r}} = \mathbf{V}_G = 0$ , and thus  $J_{\dot{r}} = J_G = 0$ . We are thus in the framework of the application of Andoyer criterion and we can conclude that the change of variables

$$(\dot{\mathbf{r}}; \mathbf{r}) \rightarrow (\dot{r}, G, G \cos i; r, w, \Omega) \quad (12)$$

is canonical. In these new variables, known as the Hill variables (Hill, 1913), the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}(\dot{r}^2 + \frac{G^2}{r^2}) - \frac{\mu}{r}. \quad (13)$$

#### 4 Delaunay variables

If we denote by  $M$  the mean anomaly,  $a$  the semi-major axis,  $e$  the eccentricity, the Delaunay variables are

$$\begin{aligned} L &= \sqrt{\mu a}, & M, \\ G &= L\sqrt{1-e^2}, & \omega, \\ H &= G \cos i; & \Omega. \end{aligned} \quad (14)$$

We define the mean anomaly  $M$  as the angle, null at perihelion, with  $dM/dt = n$ , where  $n$  is the mean motion defined as  $n^2 a^3 = \mu$ . The variable  $L$  was introduced by Delaunay (1860). As  $\mathcal{H} = -\mu/2a$ , it is the only<sup>3</sup> function  $L(a)$  such that

$$\frac{dM}{dt} = n = \frac{d\mathcal{H}}{dL} = \mu^{1/2} a^{-3/2}. \quad (15)$$

which implies

$$\frac{dL}{da} = \frac{\sqrt{\mu}}{2\sqrt{a}}, \quad (16)$$

and thus  $L = \sqrt{\mu a}$ , up to an additive constant. Following the previous section, it can be also remarked that we have  $J_\omega = G$ , and thus  $G$  is naturally conjugated to  $\omega$ , while  $(H, \Omega)$  are unchanged. After these heuristic considerations, we will formally demonstrate that the transformation from the Hill variables  $(\dot{r}, G, H; r, w, \Omega)$  to the Delaunay variables  $(L, G, H; M, \omega, \Omega)$  is canonical. As the couple of variables  $(H, \Omega)$  are unchanged, it is sufficient to consider the planar transformation  $(\dot{r}, G; r, w) \longrightarrow (L, G; M, \omega)$ . We show here the invariance of the canonical 2-form, which we find more direct than other published methods based on generating functions (Andoyer, 1913; Deprit, 1981; Floria, 1995). We need to show that

$$dr \wedge d\dot{r} + dw \wedge dG = dM \wedge dL + d\omega \wedge dG. \quad (17)$$

that is, as  $w = v + \omega$ ,

$$dr \wedge d\dot{r} + dv \wedge dG - dM \wedge dL = 0. \quad (18)$$

Using the expression of the energy in both sets of variables, we have

$$-\frac{\mu^2}{2L^2} = \frac{1}{2} \left( \dot{r}^2 + \frac{G^2}{r^2} \right) - \frac{\mu}{r} \quad (19)$$

Differentiating these expressions and taking the exterior product with  $dr$  gives

$$dr \wedge d\dot{r} = -\frac{G}{r^2 \dot{r}} dr \wedge dG + \frac{\mu^2}{L^3 \dot{r}} dr \wedge dL.$$

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<sup>3</sup> up to an additive constant

that is, as  $G = r^2\dot{v}$ ,  $dr/\dot{r} = dt$ , and  $\mu^2/L^3 = n$  where  $n$  is the mean motion,

$$dr \wedge d\dot{r} = -dv \wedge dG + ndt \wedge dL .$$

From  $dM = ndt$ , the conclusion follows.

## 5 Andoyer derivation of Delaunay variables

Eventually, one would like to derive the Delaunay variables directly, without using the intermediary Hill variables. If such a direct transform were possible, the 1-form  $\sum pdq$  should be conserved in this change of variable, and thus should be conserved in the change from the Hill variables to the Delaunay variables, that is

$$\dot{r}dr + Gdv = LdM , \quad (20)$$

which implies

$$\dot{r}^2 + G\dot{v} = \frac{1}{2}\dot{\mathbf{r}}^2 = Ln = \frac{\mu}{a} , \quad (21)$$

which is obviously wrong, as  $\dot{\mathbf{r}}^2$  is not constant in the Kepler problem. The direct Andoyer derivation of the Delaunay variables is thus not possible.

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